Lab 7 Prelab

Autonomy

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Introduction

In this lab, you will learn about Proportional-Integral-Derivative (PID) control, one of the most widely used control techniques in industry. PID controllers are found in a variety of applications ranging from simple temperature control to complex motion systems in robotics. By the end of this lab, you will understand how each component of the PID controller affects system behavior and how to properly tune the parameters for optimal performance.

1 Theoretical Background

1.1 PID Control Fundamentals

A PID controller continuously calculates an error value $e(t)$ as the difference between a desired setpoint (SP) and a measured process variable (PV), and applies a correction based on proportional, integral, and derivative terms. The mathematical form of a PID controller is:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \frac{de(t)}{dt} \tag{1}$$

Where: $u(t)$ is the control signal, $e(t)$ is the error (setpoint - measured value), K_p is the proportional gain, K_i is the integral gain, K_d is the derivative gain

1.2 The Role of Each Component

1.2.1 Proportional Term (K_p)

The proportional term produces an output proportional to the current error value:

$$P_{out} = K_p e(t) \tag{2}$$

Effect:

- Increases system responsiveness
- Reduces rise time (time to reach setpoint)
- Too high: causes overshoot and oscillation
- Too low: makes the system sluggish and unable to reach setpoint

1.2.2 Integral Term (K_i)

The integral term accumulates error over time and helps eliminate steady-state error:

$$I_{out} = K_i \int_0^t e(\tau) d\tau \tag{3}$$

Effect:

- Eliminates steady-state error
- Helps overcome system biases
- Too high: causes overshoot and oscillation
- Too low: steady-state error may persist

1.2.3 Derivative Term (K_d)

The derivative term predicts system behavior by calculating the rate of change of error:

$$D_{out} = K_d \frac{de(t)}{dt} \tag{4}$$

Effect:

- Reduces overshoot and settling time
- Improves stability
- Too high: amplifies noise and causes instability
- Too low: system may exhibit excessive overshoot

1.3 Double Integrator System

A double integrator system represents a basic model for many physical systems, like a mass moving in a frictionless environment. Its mathematical representation is:

$$\frac{d^2x}{dt^2} = u \tag{5}$$

Where x is the position, and u is the control input (force). In state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{6}$$

Where x_1 is the position and x_2 is the velocity.

1.4 Rocket Model (Double Integrator with Gravity)

The rocket model extends the double integrator by adding the effect of gravity, which creates a constant external force. This model represents systems like a vertical rocket or an elevator, where gravity consistently affects the dynamics. Its mathematical representation is:

$$\frac{d^2x}{dt^2} = u - g \tag{7}$$

Where x is the position, u is the control input (force), and g is the gravitational acceleration. In state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -g \end{bmatrix} \tag{8}$$

The key difference from the simple double integrator is that the constant gravitational force creates a bias that must be compensated for by the controller. This makes the integral term (K_i) particularly important, as it accumulates error over time and provides the necessary constant control input to counteract gravity.

1.5 Pendulum System

The pendulum is described by the nonlinear differential equation:

$$ml^2\ddot{\theta} = mgl\sin\theta - b\dot{\theta} + u \tag{9}$$

Where: θ is the angle from the upright position (0 is upright, π is downward), m is the mass at the end of the pendulum, l is the length of the pendulum, g is the gravitational acceleration, b is the damping coefficient, u is the control torque

This system is nonlinear due to the $\sin\theta$ term, making control more challenging. Note that in this formulation, the gravity term has a positive sign, as it acts to increase the angle away from the upright position.

1.6 Linear Quadratic Regulator (LQR)

The Linear Quadratic Regulator (LQR) is an optimal control technique that provides a systematic way to design feedback controllers for linear systems. Unlike PID control, which requires manual tuning, LQR determines control gains mathematically to minimize a specified performance index.

1.6.1 LQR Formulation

For a linear time-invariant system represented in state-space form:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{10}$$

The LQR controller aims to minimize the quadratic cost function:

$$J = \int_0^\infty (\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u})dt \tag{11}$$

Where: $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the control input vector, Q is a positive semi-definite matrix that penalizes state deviations, R is a positive definite matrix that penalizes control effort

The optimal control law is given by:

$$\mathbf{u} = -K\mathbf{x} = -R^{-1}B^{T}P\mathbf{x} \tag{12}$$

Where P is the solution to the algebraic Riccati equation:

$$A^{T}P + PA - PBR^{-1}B^{T}P + Q = 0 \tag{13}$$

1.6.2 Optimality of LQR

LQR is provably optimal. For the defined cost function, no other linear controller can achieve a lower cost. It provides an optimal balance between control effort and state regulation.

2 References

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3. Dorf, R. C., & Bishop, R. H. (2011). Modern control systems (12th ed.). Pearson.